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Technical Notes

Buckling Mosaic of Concentrically Hinged or Cracked Circular Plates on Elastic Foundation

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Nomenclature

b	=	normalized radius of hinge or crack
D	=	flexural rigidity
i	=	$\sqrt{-1}$
J_n	=	Bessel function of the first kind
k	=	foundation spring constant
N	=	compressive edge load
n	=	number of nodal diameters
R	=	radius of plate
r	=	radial coordinate
w	=	lateral deflection
Y_n	=	Bessel function of the second kind
γ	=	$R(k/D)^{1/4}$ stiffness parameter
θ	=	angle coordinate
λ	=	$R\sqrt{N/D}$ buckling load parameter
ν	=	Poisson ratio, taken to be 0.3

Introduction

THE buckling of thin elastic plates is extremely important in the design of load-bearing panels, especially in aircraft. The solutions for the compressive buckling load for various edge conditions may be found in [1–3]. There is less literature for a plate that is supported laterally, that is, on an elastic foundation. For rectangular plates, analytical solutions exist if two opposite sides are simply supported [4,5], otherwise, numerical means are necessary [6]. Most reports on circular plates assumed axisymmetric buckling mode [7–10], but Wang [11] recently showed nonaxisymmetric modes may sometimes give the true buckling load.

On the other hand, the rectangular plate with an internal hinge was studied by Xiang et al. [12] and the circular plate was studied by Wang [13]. Both sources did not consider foundation support.

The present Note considers the effect of an interior concentric hinge on the buckling of a circular plate on an elastic foundation. The effective hinge may be due to a closed hatch or opening, or due to a through crack. Depending on the location of the hinge and the stiffness of the foundation, the buckling force and the corresponding buckling mode are determined.

Formulation

The governing equation for the buckling of a thin, uniform elastic plate on a linear foundation is

$$D\nabla^4 w + N\nabla^2 w + kw = 0 \tag{1}$$

where D is the flexural rigidity, w is the lateral displacement, N is the compressive edge load, and k is the foundation spring constant. Normalize lengths by the radius of the plate R, and Eq. (1) becomes

$$\nabla^4 w + \lambda^2\nabla^2 w + \gamma^4 w = 0 \tag{2}$$

where $\lambda^2 = NR^2/D$ is the load parameter and $\gamma^4 = kR^4/D$ is the stiffness parameter. In polar coordinates (r, θ), let the displacement be separated as

$$w = u(r)\cos(n\theta) \tag{3}$$

where n is the number of nodal diameters. Then,

$$L^2 u + \lambda^2 L u + \gamma^4 u = 0 \tag{4}$$

$$L \equiv \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \tag{5}$$

Consider a circular plate supported on an elastic foundation. The plate has a concentric internal hinge or through crack of radius bR, $b < 1$ (Fig. 1). The plate is separated into two regions. Region I is the annular outer region $b \le r \le 1$ and region II is the inner region $0 \le r \le b$. Let subscripts I and II denote the respective regions. The bounded general solution to Eq. (4) is as follows:

1) If $\lambda > \sqrt{2}\gamma$,

$$u_I(r) = C_1 J_n(\alpha r) + C_2 Y_n(\alpha r) + C_3 J_n(\beta r) + C_4 Y_n(\beta r) \tag{6}$$

$$u_{II}(r) = C_5 J_n(\alpha r) + C_6 J_n(\beta r) \tag{7}$$

where J_n, Y_n are Bessel functions and

$$\alpha = \sqrt{\frac{\lambda^2 + \sqrt{\lambda^4 - 4\gamma^4}}{2}}, \qquad \beta = \sqrt{\frac{\lambda^2 - \sqrt{\lambda^4 - 4\gamma^4}}{2}} \tag{8}$$

2) If $\lambda = \sqrt{2}\gamma$,

$$u_I(r) = C_1 J_n(\gamma r) + C_2 Y_n(\gamma r) + C_3 r J_{n+1}(\gamma r) + C_4 r Y_{n+1}(\gamma r) \tag{9}$$

$$u_{II}(r) = C_5 J_n(\gamma r) + C_6 r J_{n+1}(\gamma r) \tag{10}$$

3) If $\lambda < \sqrt{2}\gamma$,

$$u_I(r) = C_1 Re[J_n(i\delta r)] + C_2 Re[Y_n(i\delta r)] + C_3 Im[J_n(i\delta r)] + C_4 Im[Y_n(i\delta r)] \tag{11}$$

$$u_{II}(r) = C_5 Re[J_n(i\delta r)] + C_6 Im[J_n(i\delta r)] \tag{12}$$

Here, $i = \sqrt{-1}$ and

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$$\delta = \sqrt{\frac{-\lambda^2 + \sqrt{4\gamma^4 - \lambda^4 i}}{2}} \quad (13)$$

If the outer edge is clamped, displacement and slope are zero:

$$u_I(1) = 0, \qquad \frac{du_I}{dr}(1) = 0 \quad (14)$$

Note that, for region II, the Bessel functions of the second kind are absent due to the finiteness of the displacement at the origin. If the outer edge is simply supported, displacement and moment are zero:

$$u_I(1) = 0, \qquad \frac{d^2 u_I}{dr^2}(1) + \nu\left[\frac{du_I}{dr}(1) - n^2 u_I(1)\right] = 0 \quad (15)$$

The matching conditions are continuity of displacement and shear at $r = b$

$$u_I(b) = u_{\mathrm{II}}(b) \quad (16)$$

$$\frac{d^3 u_I}{dr^3}(b) + \frac{1}{b}\frac{d^2 u_I}{dr^2}(b) - \left\{\frac{[1 + n^2(2-\nu)]}{b^2} - \lambda^2\right\}\frac{du_I}{dr}(b) + \frac{n^2(3-\nu)}{b^3}u_I(b) = \frac{d^3 u_{\mathrm{II}}}{dr^3}(b) + \frac{1}{b}\frac{d^2 u_{\mathrm{II}}}{dr^2}(b) - \left\{\frac{[1 + n^2(2-\nu)]}{b^2} - \lambda^2\right\}\frac{d^2 u_{\mathrm{II}}}{dr}(b) + \frac{n^2(3-\nu)}{b^3}u_{\mathrm{II}}(b) \quad (17)$$

Here, the terms containing λ represent the shear component of the edge force [14]. The moments are also zero at the hinge:

$$\frac{d^2 u_I}{dr^2}(b) + \frac{\nu}{b^2}\left[b\frac{du_I}{dr}(b) - n^2 u_I(b)\right] = 0 \quad (18)$$

$$\frac{d^2 u_{\mathrm{II}}}{dr^2}(b) + \frac{\nu}{b^2}\left[b\frac{du_{\mathrm{II}}}{dr}(b) - n^2 u_{\mathrm{II}}(b)\right] = 0 \quad (19)$$

Upon substitution of u_I, u_{II}, Eqs. (14) and (16–19) or Eqs. (15–19) represent a set of six algebraic equations with six unknowns C_j. For nontrivial solutions, the determinant of the coefficients is set to zero, yielding the eigenvalues λ for each given n and stiffness γ. The lowest λ is the square root of the normalized buckling force. The determinant is exact, and the roots for λ can be obtained to any accuracy by a bisection method.

Results and Discussions

First consider the case where the outer boundary is clamped. Figure 2 shows the buckling load versus the radius of the internal hinge for various constant foundation stiffnesses. Such a "buckling mosaic" is much more informative than listing tables. If the hinge is absent ($b = 0$), our solution agrees with that of [11], where the buckling modes alternate between $n = 0$ and $n = 1$. When the foundation is absent ($\gamma = 0$), our results compare well with the axisymmetric buckling of [13] when the hinge does not have rotational resistance. However, [13] failed to discover a segment of asymmetric buckling ($n = 1$) between $b = 0.451$ ($\lambda = 3.44$) and

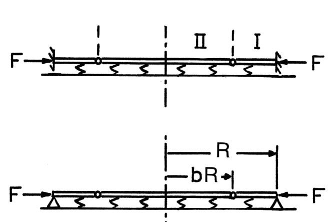

Fig. 1 Cross section of the circular plate (top: clamped case; bottom: simply supported case). Small circles represent the location of the hinge or crack.

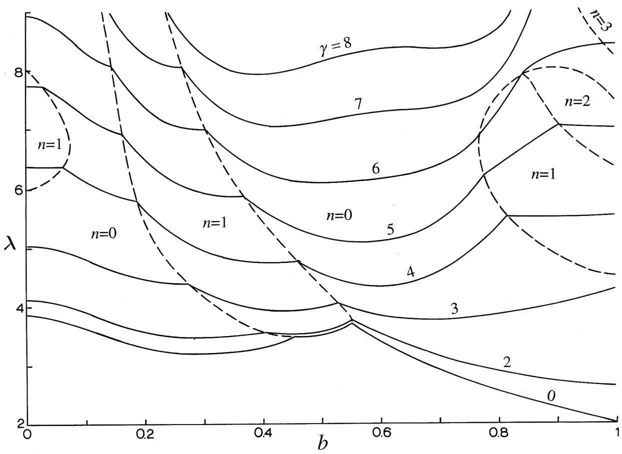

Fig. 2 Buckling load versus hinge location for various constant foundation stiffness. The outer edge is clamped. Dashed lines show boundaries of mode changes.

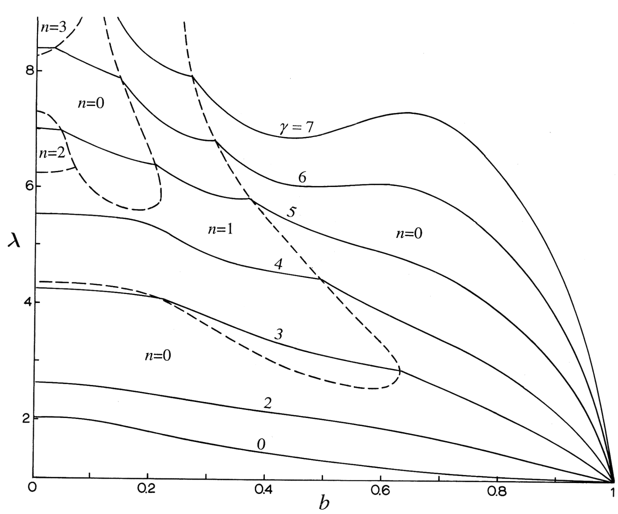

Fig. 3 Buckling load versus hinge location for various constant foundation stiffness. The outer edge is simply supported. Dashed lines show boundaries of mode changes.

$b = 0.554$ ($\lambda = 3.69$) which gives the true buckling load. The situation is more complicated if a foundation is present. Aside from the middle swath of $n = 1$ modes, for small b and large b, pockets of $n = 1$ mode appear, the latter joined by a region of $n = 2$ mode. For larger foundation stiffness, higher modes give the buckling loads, such as the $n = 3$ mode in the right corner of the figure.

Figure 3 shows the case when the outer boundary is simply supported. When the foundation is absent, our results agree with [13]. When the hinge is absent ($b = 0$), the buckling loads agree with [11], except for pockets of higher modes missed by [11]. The $n = 2$ buckling mode occurs between $\gamma = 4.445$ ($\lambda = 6.296$) and $\gamma = 5.319$ ($\lambda = 7.427$), and the $n = 3$ buckling mode occurs between $\gamma = 5.946$ ($\lambda = 8.360$) and $\gamma = 6.687$ ($\lambda = 9.395$). For the simply supported case, it is safe to say that the axisymmetric ($n = 0$) buckling mode would happen if $b > 0.64$ or $\gamma < 2.52$. All curves decrease to the thin annulus limit [14] of $\lambda = \sqrt{1 - \nu^2} = 0.9539$ as $b \to 1$.

Because of an elastic foundation, $\gamma \neq 0$. Thus, neither α, β, λ, nor δ would be zero and the Bessel functions do not degenerate. Unlike unsupported cracked plates, no rigid deformations are possible.

Conclusions

The buckling loads (and the corresponding modes) of a hinged or cracked circular plate on an elastic foundation are determined. These results are exact, and serve as standards for other numerical schemes. The interaction of the hinge with foundation support is found to be complex, as shown in our buckling mosaics. As stiffness is increased, higher buckling modes are present, including some not discovered in the special cases of [11] or [13].

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